Existence and Uniqueness Results for Two-Dimensional Stochastic Linearised Boussinesq Equation

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Abstract

The water flow in saturated zones of the soil is described by two-dimensional Boussinesq equation. This paper is devoted to investigating the linearised stochastic Boussinesq problem in the presence of randomness in hydraulic conductivity, drainable porosity, recharge, evapotranspiration, initial condition and boundary condition. We use the Sabolev spaces and Galerkin method. Under some suitable assumptions, we prove the existence and uniqueness results, as well as, the continuous dependence on the data for the solution of linearised stochastic Boussinesq problem. Keywords: linearised stochastic Boussinesq equation, Galerkin method, existence and uniqueness results, and continuous dependence on the data.

Keywords: existence and uniqueness results for two-dimensional stochastic linearised Boussinesq equation

1. Introduction

The water flow for an unconfined aquifer of the soil is described by Boussinesq equation (1904). The equation results from the application of the mass conservation principle, Darcy's law, and the Dupuit-Forchheimer hypothesis (Bear, 1972). The two-dimensional Boussinesq equation is:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_{\mathrm{i}}} [\frac{k_{\mathrm{i}}}{s}(u+d)\frac{\partial}{\partial x_{\mathrm{i}}}(u+d)] + \frac{r-ET}{s}, \quad (1)$$

$(x,t) \in D \times (0,T]$,

where $x=(x_1,x_2) \in D \subset R^2$ denote the space coordinates (variables), t is the time variable, D denotes a bounded spatial domain with smooth boundary $\partial D,\ T > 0$ is a constant,

$x = (x_1,x_2),\ \ |x|^2 = x^2{}_1 + x^2{}_2,\ \ dx = dx_1 * dx_2,\ u = u(x,t)$ represents the elevation of the free surface (or hydraulic head) above the impervious layer, d denotes aquifer's depth (measured from the impervious layer), k_{i} denotes the saturated hydraulic conductivity of the soils along i direction, ET denotes the evapotranspiration, r denotes the recharge, s denotes the drainable porosity.

The Boussinesq equation associated with initial condition:

u (x, 0) = u_0(x) , x$\in$ D, (2)

and with Dirichlet boundary condition

$u\ (x,t) = H\ (x,t), \quad (x,t) \in \partial D \times (0,\mathrm{T})$ (3)

where $u_0\ (x)$ and $H\ (x,t)$ are given function. Similarly, we can consider the Neumann boundary condition.

$$\frac{\partial u(x,t)}{\partial u\ (x)} = H_1(x,t) \quad , \quad (x,t) \in \partial D \times (0,T]$$

The linearised Boussinesq equation is

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}[\frac{k_i}{s}H_0\frac{\partial}{\partial x_i}(u+d)]+\frac{r-ET}{s},\ \ (x,t)\in D\times(0,T]\ \ (4)$$

where H_0 (constant) represent the average depth of the aquifer.

In the present study we investigate the linearised stochastic Boussinesq equation

$$\frac{\partial u(x,t,\omega)}{\partial t}=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}[\frac{k_i(x,t,\omega)}{s(t,\omega)}H_0\frac{\partial}{\partial x_i}(u(x,t,\omega)+d(x,\omega))]+\frac{r(x,t,u,\omega)-ET(x,t,u,\omega)}{s(t,\omega)}$$

$$(x,t,\omega)\in D\times(0,T]\times\Omega \qquad (5)$$

Where $\Omega=\{\omega\}$ denotes the sample space and ω denotes the probabilitary variable.

The equation (5) associated with initial condition

$$u(x,0,\omega)=u_0(x,\omega)\ \ ,\ \ (x,\omega)\in D\times\Omega \qquad (6)$$

and with Dirichlet boundary condition

$$u(x,t,\omega)=H(x,t,\omega)\ \ ,\ \ (x,t,\omega)\in\partial D\times(0,T]\times\Omega \quad (7)$$

Where $u_0(x,\omega)$ and $H(x,t,\omega)$ are given random function and random field, respectively.

Similarly, we can consider von Neumann boundary condition.

The Modern Soil science considers the soil as a complex dynamical system which evolves under total impact of the interactions between natural and biological factors as well as the human activity. Because these interactions are random processes, it is reasonable to assume that the water flow in soil is described by a random field. Measurement errors of hydraulic conductivity, drainable porosity, recharge and evapotranspiration are the other reason for the presence of randomness in the soil's water flow process. These arguments have a lot of supporters among the soil scientists and physicists, who are showing a greater interest for stochastic models of water flow in soil. They use the stochastic models effectively in their work (research), see Freeze (1975), Cordova and Bras (1981), Chung and Austin (1987), etc.

The experimental data from many countries, including Albania, support the hypothesis that hydraulic conductivity of the saturated soil, drainable porosity, recharge and evapotranspiration are random variables, see Averjanov (1972), Van Schilfgaarde (1974-1979), Freeze (1975), Hubert (1976), Skaggs and Tang (1976) , Sagar and Preller (1980), Cordova and Bras (1981), Kolaneci, Xinxo and Bica (1983), Chung and Austin (1987), etc.

Depending on how the randomness is present (introduced) in unsteady water flow in soil, there are four mathematical problems, with increasing level of complexity:

1. The problem with random initial condition.
2. The problem with random boundary condition.
3. The problem with random recharge or evapotranspiration.
4. The problem with random hydraulic conductivity or drainable porosity.

The important achievements in the study of stochastic linearised Boussinesq problem are:

The paper is organized as follows. In section 2 we formulate the problem (5), (6), (7) in an appropriate functional setting. In section 3 we give the existence and uniqueness theorem for problem (5), (6), (7). Section 4 contains conclusion.

2. Functional setting and formulation of the problem.

Let (Ω,F,μ) or (Ω,F,P) be a complete probability space, where $\Omega=\{\omega\}$ denotes the space of elementary events (or the space of basic outcomes), F is the σ-algebra associated with Ω, and μ (or P) is the probability measure defined on F.

The σ -algebra F can be interpreted as a collection of all random events $\omega \in \Omega$ and that have a well-defined probability with respect to F. In the present study use the Theory of Sobolev Spaces, see Adams (1975), Tribel (1986).

A real-valued random variable X=X(ω) is a mapping $X: \Omega -> R$.

Assume that the probability measure μ has a numerable basis D⊂ R^2 denotes a bounded domain with smooth boundary $\partial D,\ 0 < T < +\infty \quad Q_T = D \times (0,T]$ and $S_T = \partial D \times (0,T]$

The separable Hilbert space $L^2(\Omega)$ is well-known

M= {f (x,ω): $D \rightarrow L^2(\Omega)$} denotes the set of second order random functions f(x,ω) over the domain$D$.

Define the space

H= L^2 (D; L^2 (Ω))= {f= f (x, ω) ∈ M,

$|| f ||_\Omega \in L^2$ (D)},

Equipped with scalar product

$(f,g)_H = \int_D (f,g)_\Omega\, dx, \forall f, g \in$ H.

The induced norm from the scalar product is:

$|| f ||_H = (\int_D (|| f ||_\Omega{}^2\, dx)^{1/2}, \forall f \in$ H.

Proved that H is separable Hilbert.

Definition of the functional spaces C^∞(D),

$\mathcal{L}$(D), $\mathfrak{D}^1(D; L^2(\Omega)),\ \ C^m(D; L^2(\Omega)) for\ m \geq 0, C_c^m(D, L^2(\Omega))\ and\ H^m = H^m(D; L^2(\Omega))$ are well known.

For f= f(x,ω)∈ M are defined generalised derivates of order α with respect to x:

$D^\alpha f(\varphi) = (-1)^{|\alpha|} \int_D f\, D^\alpha \varphi dx, \varphi \in \mathfrak{D}(D),\ \alpha = (\alpha_1, \alpha_2), |\alpha| = \alpha_1 + \alpha_2$).

Denotes the outward normal in an arbitrary point x∈ ∂D.

if f∈ $H^m = H^m(D; L^2(\Omega))$, ≪m- trace of f on ∂D ≫ is defined:

$\gamma^{(m)}(f) = \{f, \frac{\partial f}{\partial}, \ldots, \frac{\partial^{m-1} f}{\partial^{m-1}}\}$, m=2,3,4,...

It is obvious that $\gamma^{(m)}(f) \in (L^2(D); L^2(\Omega))^m$, where the right-hand side denotes Cartesian product.

Consider the space.

$H_0^m = H_0^m(D, L^2(\Omega))$= {f∈ $H^m, \gamma^{(m)}(f) = 0\}$ $\quad H_0^m$ is a closed subspace of H^m.

The Banach space $L^\infty(\Omega)$,$L^\infty(Q_T; L^2(\Omega))$ and $L^\infty((Q_T; L^\infty(\Omega))$ are well known.

Consider the separable Hilbert space V such that V⊂H, V dense in H, V⊂ H (V continuously embedded in H).

That is, (V, H, V') represent a Gelfand triplet.

$(\ ,\ )_V$denotes the scalar product in V,

$(\ ,\ )_H$denotes the scalar product in H,

$(\ ,\ )_{V^1}$denotes the scalar product in V′,

The induced norms are $||\ ||_V, ||\ ||_H, ||\ ||_{V'}$

Respectively.

Defined the space

$$L^2(0,T,V) = \{f = f(x,t,\omega): [0,T] \to V, \quad \int_0^T \|f\|_V^2 dt < +\infty\}$$

Equipped with scalar product

$$(f,g)_{L^2(0,T,V)} = \int_0^T (f,g)_V dt \quad \forall f,g \in L^2(0,T,V)$$

The induced norm from the scalar product is

$$\|f\|_{L^2(0,T,V)} = (\int_0^T \|f\|_V^2 dt)^{1/2}$$

$L^2(0,T,V)$ is separable Hilbert space.

Similarly defined the separable Hilbert space$L^2(0,T,H)$, $L^2(0,T,V')$, $L^2(-\infty,T,V)$ and $L^2(-\infty,T,H)$, see Tribel (1986).

Defined the space

$$W(0,T) = \{f \in L^2(0,T,V)\,, D_t f \in L^2(0,T,V')$$

Equipped with scalar product

$$(f,g)_w = \int_0^T [(f,g)_V + (D_t f, D_t g)_{V'}]dt \quad \forall f,g \in W(0,T)$$

The induced norm in $W(0,T)$ from the scalar product is

$$\|f\|_W = (\int_0^T [\|f\|_V^2 + \|D_t f\|_{V'}^2]dt)^{1/2}$$

$W(0,T)$is separable Hilbert space $W(0,T) \subset C([0,T],H)$,

$< u'(t), v >= D_t(u(t),v)_H = (u'(t),v)_H$,

$$\forall\, u(t) \in W(0,T) \; and \; \forall\, v \in V$$

$$D_t\|u(t)\|_H^2 = 2 < u'(t), u(t) >, \forall\, u(t) \in W(0,T)$$

see Dautary and Lions (1985), Tribel (1986).

The symbol $<.,.>$ denotes the duality between the Hilbert spaces V' and V.

Given a family $a(t,u,v)$ of the continous bilinear forms defined on V× V with the parameter $t \in (0,T)$.

Assume that $a(t,u,v)$ satisface the conditions:

$\exists \alpha$ positive constant real number $c = c(T)$ with that

$|a(t,u,v)| \leq c\|u\|_V \, \|v\|_V$, $\forall t \in (0,T)$, $\forall u \in V \; and \; \forall v \in V$ (8)

$\exists$ the constant real numbers $\mu \; and \; \mu > 0$ with that

$a(t,u,v) + \mu\|u\|^2{}_V \geq \alpha\|v\|_V{}^2$, $\forall t \in (0,T)$, $\forall v \in V$ (9)

The condition (9) is V-chercivate hypothesis

The family of random operators A(t) associated with $a(t,u,v)$ is defined by

$a(t,u,v) =< A(t)u, v >, \forall t \in (0,T),\ \forall u \in V\ and\ \forall v \in V$(10)

Is proved that A(t) ∈ $L(v,V')$ and $A(t) \in L(L^2(0,T,V); L^2(0,T,V'))$, see Lions (1972).

If $A(t)u \in H, \forall t \in (0,T),\ \forall u \in V$ then we can prove that

$< A(t)u, v >= (A(t)u,v)_H,\ \forall v \in V$ (11)

Consider the random evolutionary equation:

$D_t u(t) + A(t)u(t) = f(t)\ \ (x,t,\omega) \in Q \times \Omega$(12)

With initial condition

$u(0) = u_0,\ \ (x,\omega) \in D \times \Omega$ (13)

The family of random operators A(t) is defined by (10). Assume that $f(t) \in L^2(0,T,V') and\ u_0 \in H$

Problem 1

Given $(t) \in L^2(0,T,V')$, $u_0 \in H\ and\ A(t)$ defined by (10). Find the random field

$u = u(x,t,\omega) \in W(0,T)$ Which satisfies (12) for almost all $(x,t,\omega) \in Q \times \Omega$ and (13).

By definition, $u = u(x,t,\omega)$ is a solution of the problem (12), (13) if the above mentioned conditions are satisfied?

3. Existence and uniqueness results

In this section we will prove:

Theorem 1

Suppose that the assumptions (8), (9) are satisfied. Prove that the problem 1 has a unique solution $u = u(x,t,\omega) \in W(0,T)$ which depends continuously on the data. The proof of Theorem 1 undergoes through several steps.

Step1. Preliminary reduction of the problem

We can assume that (9) is satisfied for $\lambda = 0$. Substituting $u = z^{ekt}$ where k is an arbitrary real number, the problem 1 transformed in equivalent problem:

Find $z = z(x,t,\omega)$ which satisfies the identity $a(t,z,v) + k(z,v)_H + D_t(z,v)_H = (e^{-kt}f,v)_H + (u_0,v)_H, \forall v \in V,$

With $Z(x,t,\omega) \equiv 0$ for t<0. In this identity, $a(t,u,v)$ substituded by $a(t,z,v) + k(z,v)_H$. Choosing $k = \lambda$ we obtain the desirable result.

Step II. Proving the existence of the solution

To prove the existence of the solution of stochastic problem (12), (13), we modify the Galerkin method, developed by Dautray and Lions (1985) for deterministic parabolic partial differential equations, see Dautray and Lions (1985), pp 619-627.

The Hilbert space V is separable. Therefore, ∃ is the basis $\omega_1, \omega_2, \omega_3, \dots, \omega_m$..... Of V in the following sense: $\forall m \in N$ the elements $\omega_1, \omega_2, \omega_3, \dots, \omega_m$..... are the linearly independent and the set of the all finite linear combinations

$\sum_n \zeta_n w_n,\ \zeta_n \in R, n \in N$ is dense in V.

For each m=1,2,3,…. Define approximate solution $u_m = u_m(x,t,\omega)$ of the problem (12) ,(13) by using the following method:

$u_m = \sum_{i=1}^{m} g_{im}(t) w_i$ (14)

$(D_t u_m, w_j)_H + a(t, u_m, w_j) =< f, w_j > \quad \forall j = 1,2,3, \dots\dots, m$ (15)

$u_m(0) = u_{0m} \sum_{i=1}^{m} \zeta_{im} w_i$(16)

where u_{0m} is the orthogonal projection of $u_0 \in H$ over the subspace spanned on

$w_1, w_2, w_3, \dots., w_m$. More generally, u_{0m} denotes each element of the above mentioned subspace, which satisfies the condition:

$$\lim_{m\to+\infty} \|u_{0m} - u_0\|_H = 0$$

The system (15) with initial condition (16) represents the Cauchy problem for the unknown deterministic functions

$g_m(t) = \{g_{im}(t)\}_{i=1}^{m}$:$\{w_m D_t g_m(t) + A_m(t) g_m(t) = f_m(t)$, $g_m(0) = \{\zeta_{im}\}_{1\le i\le m}$.

The matrices are:

$$W_m = \left\|(w_i, w_j)_H\right\|_{1\le i\le j\le m} ,$$

$$A_m(t) = \left\|a(t, u_i, w_j)\right\|_{1\le i\le j\le m} ,$$

$$g_m(t) = \{g_{im}(t)\}_{i=1}^{m} \ and \ \ f_m(t) = \{f(t), w_j\}_{1\le i\le m}$$

W_m is a non degenerate matrix, because of $w_1, w_2, w_3, \dots., w_m$ are linearly independent. Therefore, the system (15), (16) has unique solution $g_m(t)$ for $t \in (0, T)$, see Arnold (1975).

$f(t) \in L^2(0, T, V')$ Implies that $[g_{im}(t)]^2$ are integrable functions.

Therefore, the

$u_m = u_m(t) = u_m(x, t, \omega) \in L^2(0, T, V')$ and $D_t u_m(t) \in L^2(0, T, V')$ (17)

The proof of Theorem 1 continues similarly to the arguments presented by Dautray and Lions (1985), pp 619—627.

We obtain the following results:

$D_t\|u_m\|_H^2 + 2\alpha\|u_m\|_V^2 \le 2 < f, u_m >\le \alpha\|u_m\|_V^2 + \frac{1}{\alpha}\|f\|_{V'}^2$, (18)

where α denotes the cohercivity constant in (9) ,

$sup\|u_m(t)\|_H^2 \le \|u_0\|_H^2 + \frac{1}{\alpha}\|f\|_{L^2(0,T,V')}^2$,

the sequence $\{u_m(t)\}$ is strogly bounded in the space $L^\infty(0, T, H)$, (19)

the sequence $\{u_m(t)\}$ is strogly bounded in the space $L^2(0, T, V)$. (20)

There exist the element $u \in L^2(0, T, V)$and the subsequence of natural numbers m', such that $u_{m'} \to u$ in $*$ weak topology of the space $L^\infty(0, T, H)$ as m'$\to +\infty$:

$\exists v \in L^1(0, T, H)$ such that

$\lim_{m'\to+\infty} \int_0^T (u_{m'} - u, v)_H dt = 0$, (21)

There exist $u_* \in L^2(0, T, V)$ and the subsequence m" of the sequence m', such that $u_{m"} \to u_*$ in weak topolgy of $L^2(0, T, V)$ as m"$\to +\infty$

$\lim_{m''\to+\infty} \int_0^T < u_{m''} - u_*, v > dt = 0$,$\forall v \in L^2(0, T, V')$ (22)

$u_* = u$ and $\lim_{m''\to+\infty} \int_0^T (u_{m''} - u_*, v)_H dt = 0$, $\forall v \in L^2(0, T, H)$, (23)

$u = u_* \in L^\infty(0,T,H) \cap L^2(0,T,V)$ (24)

$u = u_*$ is solution of the problem (12),(13). That is, u satisfies the equation (12) and the initial condition (13)

Step III. Proving the uniqueness of the solution

Assume that u and $\bar{u}$ are solutions of the problem (12), (13) with data {f, u_0} and { $\bar{f}$, $\overline{u_0}$}, respectively. Using Vishik-Ladizhenskaja lemma, we prove that

$$\|u - \bar{u}\|_{L^2(0,T,V)} \leq \frac{1}{\alpha}\|u_0 - \overline{u_0}\|_H^2 + \frac{1}{\alpha^2}\left\|f - \bar{f}\right\|_{L^2(0,T,V')}^2$$

Where α denotes the cohercivity constant, see (9). The proof of Theorem 1 is complete.

Remark 1

The appropriate boundary condition for the stochastic problem (12), (13) depends on the choice of the Hilbert space V and random field $f = f(x,t,\omega)$

Choose $V = H_0^1(D, L^2(\Omega))$and suppose that $(f,v) = \int E(f_1 v)dx + \int E(f_2 v)ds$,$\forall v \in V$ where

$f_1 \in L^2(Q_T; L^2(\Omega))$ and $f_2 \in L^2(S_T; L^2(\Omega))$

Under these conditions, the appropriate boundary condition for the problem (12), (13) is von Neumann condition

$$\frac{\partial u}{\partial v} = f_2(x,t,\omega)\,, \quad (x,t,\omega) \in S_T \times \Omega,\;\; S_T = \partial D \times (0,T)$$

Remark 2

We distinguish between the deterministic and stochastic Boussinesq Problems:

Galerikin method is an interior approximation f the Hilbert space V, see Team (1981). Therefore, in the stochastic Boussinesq problem are discredited the spatial variables x_1, x_2 as well as the probabilitary variable ω.

Conclusion

In this paper we have studied the stochastic linearised two-dimensional Boussinesq problem in the Hilbert spaces. The randomness is the present in hydraulic conductivity, drainable porosity, (recharge-evapotranspiration), initial condition and boundary condition. We use Sobolev spaces and Galerkin method. Under suitable assumptions, we prove the existence and uniqueness theorem for the solution of the problem (12), (13).Numerical approximations of the solution of stochastic linearised Boussinesq equation is the next step of the present study. One important application of our study is in agricultural drainage.

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