Eigenfrequency and Euler's Critical Load Evaluation of Transversely Cracked Beams with a Linear Variation of Widths

Matjaž Skrinar

Department of Structural Mechanics Faculty of Civil Engineering,

Transportation Engineering and Architecture University of Maribor

Abstract

For a truthful evaluation of the mechanical response of structures reliable and adequate computational models are essential. Consequently, various researches have been devoted to the mathematical representation of cracked structures. This paper studies the performance of the simplified crack model in estimations of fundamental eigenfrequency as well as elastic Euler's critical load for transversely cracked beams of rectangular cross-sections with linearly-varying widths. To obtain these solutions for different beams with diverse boundary conditions Rayleigh's energy method which requires an assumed transverse displacement function can be applied. After the appropriate displacement function is being selected, kinetic and strain energy, as well as the work done by an external axial compressive force P are evaluated. From these values, the estimations of the fundamental eigenfrequency, as well as the critical load, are assessed. To obtain these preliminary estimates, static deflection functions were applied initially. These functions represent a wide group of suitable functions since they automatically satisfy the required kinematic boundary conditions. Afterwards, alternative functions constructed from a dedicated polynomial solution were applied. Since this mathematical form offers straightforward integration, the genuinely applied displacement functions were further upgraded, separately for eigenfrequency as well as for critical load estimation. All obtained simplified model's solutions were afterwards compared to the results from equivalent and more detailed 3D finite models of the examined structures. The comparisons of the results demonstrated very fine agreements with the results from 3D FE models for all performed analyses. The considered simplified model thus clearly yields a suitable alternative in modelling of cracked beams with a linear variation of width in those situations, where cracks have to be considered within the analysis.

Keywords: Cracked beams with transverse cracks; linear cross-sections' variations; simplified computational model; transverse displacements- functions; fundamental eigenfrequency evaluation; Euler's critical load.

1. Introduction

Any degenerative effect in structures during the utilization alters their mechanical reaction by considerably decreasing the stiffness and potentially leading to their failure. Therefore, several studies consider the detection and identification of stiffness reductions in engineering structures. Such approaches for damages recognition and classification are often based on the measured structure's answer since the occurrence of damage changes the structures' response parameters. However, the efficiency of these strategies depends on the quality of measured data as well as on computational models implemented.

When analyzing cracked structures' response thorough meshes of 2D or 3D finite elements offer the finest description of a general structure, as well as of the cracks and their surroundings. Despite this, simplified models requiring fewer data are usually implemented in structural health monitoring techniques. The "discrete spring" model presented by Okamura et al. (Okamura et al., 1969) is the model that has been implemented in numerous research studies. Due to its simplicity this simplified model has been intensively applied in vibration analysis of cracked beams (F. Bakhtiari-Nejad et al., 2014), new various approaches for inverse identification of cracks (Labib et al., 2015), as well as in experimental inverse identifications of a crack (Cao et al., 2014) or a concentrated damage (Greco and Pau, 2011). Further, several papers were devoted to Euler–Bernoulli beam's finite element having an arbitrary number of transverse cracks differing in the principles of mechanics applied to obtain closed-form solutions of the genuine governing differential equation for transverse displacements (Biondi and Caddemi, 2007; Palmeri and Cicirello, 2011; Skrinar, 2009; Skrinar and Pliberšek, 2012).

The majority of the research has been limited to structural elements with constant rectangular cross-sections. Skrinar and Imamović (Skrinar and Imamović, 2018) studied bending of beams of various heights' variations along the length implementing a multi-stepped multi-cracked beam finite element (Skrinar, 2013) where the genuine continuous variation of height was modelled by an adequate series of steps. Although this model offers good (but approximate) results, it is limited to bending analyses only.

In this paper, the area of utilization of the simplified crack model is expanded to beams with linearly-varying widths where Rayleigh's energy method is being implemented for fundamental frequency and Euler's critical load estimations.

2. Simplified Computational Model

As a crack in a beam alters the local compliance, the crack is in Okamura "discrete spring" mathematical representation modelled as a massless rotational linear spring of appropriate stiffness. The neighbouring non-cracked parts of the beam to the left and to the right of the crack are modelled as elastic elements, connected by a linear spring. For the first definition for rotational spring's stiffness (given by Okamura et al. for a rectangular cross-section) as well as for all other definitions, the linear moment-rotation constitutive law is adopted. The model thus allows for a rather effortlessness description of a crack as only two parameters are required: its location L_1 from the left end of the beam, and its depth which governs the spring stiffness K_r.

3 Implementation of Rayleigh's Energy Method for Fundamental Frequency and Euler's Critical Load Estimations

Structural analysis is mainly concerned with the determination of a physical structure's response when subjected to some action. Each new computational model's behaviour must be therefore tested in various engineering situations such as static, dynamic or buckling analyses.

In dynamic analysis, eigen or natural frequencies are one of the basic properties of elastic dynamic systems. Each such system has one or more natural frequencies i.e. frequency at which it tends to vibrate freely in the absence of any driving or damping force. Therefore, the simplified model's abilities in the first natural frequency prediction were studied as they dependent only on the structure's properties (its stiffness and participating mass) and not on the load function.

There are many available methods for determining the natural frequency (Newton's Law of Motion, Rayleigh's Method,...). Some of these methods yield a governing equation of motion (from which the natural frequency may be determined afterwards), and the others produce the natural frequency only.

In this study, which examines the behaviour of the simplified computational model, Rayleigh's method (also known as the energy method), which reduces the dynamic system to a single-degree-of-freedom system consequently yielding just the first natural frequency is being utilised.

Rayleigh's method requires an assumed displacement function *w(x)*. If this function is identical to the solution of the corresponding differential equation of motion (i.e. mode shape), the true fundamental frequency is being obtained. As this is seldom true, the assumed displacement function introduces additional constraints. Because they increase the stiffness of the system, Rayleigh's method overestimates the true fundamental frequency. The fundamental lemma of the method thus states that the total energy of the system is equal to the maximum kinetic energy which also equals the maximum deformation (potential) energy.

For the situation where the breathing of the crack is not considered i.e., the crack remains open, the "strain" energy (the potential energy stored as elastic deformation of the structure including crack) is approximated as:

$$U_{strain} = \frac{1}{2} \cdot \left(\int_{x=0}^{L_1} EI(x) \cdot (w_1''(x))^2 \cdot dx + \int_{x=L_1}^{L} EI(x) \cdot (w_2''(x))^2 \cdot dx + K_r \cdot (w_1'(L_1) - w_2'(L_1))^2 \right) \quad (1)$$

In Eq.(1) functions $w_1(x)$ and $w_2(x)$ are functions that represent the transverse displacements and must satisfy the most important kinematical boundary conditions, such as displacement and rotation. The more accurate displacement function also provides a more accurate result. In the absence of an exact solution of the differential equation (mode shape), approximate functions are applied, where static deflection functions *v(x)* represent a wide group of suitable functions since they automatically satisfy the required kinematic boundary conditions.

To obtain the first eigenfrequency estimation the beam's kinetic energy is approximated as

$$U_{kin} = \frac{\omega_1^2}{2} \cdot \left(\int_{x=0}^{L_1} m(x) \cdot (w_1(x))^2 \cdot dx + \int_{x=L_1}^{L} m(x) \cdot (w_2(x))^2 \cdot dx \right) \quad (2)$$

implementing the same displacement functions.

Afterwards, the first in-plane vibrations eigenfrequency estimate is obtained from the total mechanical energy conservation law:

$$U_{strain} = U_{kin} \quad (3)$$

The results from these functions *w(x)* can be improved by evaluating new upgraded displacement functions due to a transverse load, given as $q(x)=m(x)\cdot\omega^2\cdot w(x)$.

The strain energy approximation can be also applied in the energy method for the buckling load evaluation. The method assumes that the elastic system is a conservative system in which energy is not dissipated as heat, and, therefore, the energy added to the system by the applied external forces is stored in the system in the form of strain energy. The work (i.e. "applied" energy) done on the system by an external axial compressive force P is evaluated by applying the same transverse displacements functions:

$$U_{app} = \frac{P}{2} \cdot \left(\int_{x=0}^{L_1} (w_1'(x))^2 \cdot dx + \int_{x=L_1}^{L} (w_2'(x))^2 \cdot dx \right) \quad (4)$$

The energy conservation law states:

$$U_{strain} = U_{app} \quad (5)$$

from which the estimate of the buckling load P_{crit} can be evaluated.

Therefore, although the same static transverse displacements function due to bending allow for a very straightforward implementation either in natural frequency as well as in buckling analysis, their solutions are not the finest.

Therefore, the assumed displacement functions *w(x)* are usually constructed from the analysed problem's dedicated polynomial solution, primarily due to ease of their integration which is essential for a successful subsequent upgrade of the solutions. Among the results obtained by implementing various assumed displacement functions, the smallest value yields an upper limit of the true fundamental frequency or buckling load.

4. Numerical Validations

Four cracked fundamental beam-structures were analyzed in order to investigate the effectiveness of the simplified model. For all four structures that differed only in boundary conditions, the length *L* was 10 m and the Young modulus was 30 GPa with Poisson's ratio 0.3. The cross-section was a rectangle with height *h* = 0.2 m where the width *b* was linearly increasing from 0.1 m at the left-end to 0.2 m at the right-end. A single transverse crack of the depth of 0.1 m was located at the mid-span to maximise its impact on the results for the majority of the examples, and the rotational spring's definition given by Okamura was selected.

The obtained results were further compared with the values from a commercial finite element program COSMOS/M where corresponding 3D finite models of the considered structures were established and analyzed. The computational model consisted of 48,000 3D solid finite elements with almost 75,000 nodal points. In each node, three degrees of freedom were taken into account – vertical and two horizontal displacements. The model's vertical and horizontal displacements were obtained in discrete points by solving more than 220,000 linear equations. Since this model allows for a realistic description of the crack those results further served as the reference values.

In the first phase, the first eigenfrequency and the buckling load estimations were obtained by implementing static deflection functions due to a downward vertical uniform load q=2000 N/m along the complete structure. These functions were further introduced into Eqs.(1)-(5).

Afterwards, basic polynomial functions were constructed for each of the considered structures considering general boundary conditions only. The implementation of these functions in Eqs.(1)-(5) yielded new sets of results for the first eigenfrequency and the buckling load.

These basic general polynomial functions were also upgraded accordingly to the specific problem to see the impact of functions' improvement to the quality of the results for both studied problems.

In the penultimate step, special polynomial functions devoted exclusively to buckling analyses were created by considering additional boundary data. Ultimately, also these functions were upgraded.

4.1 Simply Supported Beam

Initially, the governing differential equation of the elastic line for a slender beam subjected to bending in the plane of symmetry was solved. This equation, known also as Euler–Bernoulli equation of bending, relates transverse displacement *v(x)*, the coordinate *x*, the geometrical and mechanical properties of the cross-section (unified in flexural rigidity *EI(x)*), and the applied transverse load *q(x)*. For the case considered where the flexural rigidity *EI(x)* is not a constant value this relation yields a fourth-order ordinary differential-equation with non-constant coefficients. However, the crack, located arbitrarily within the beam ($0 \leq L_1 \leq L$), separates the beam into two elastic parts, and to obtain the transverse displacements two coupled differential equations had to be solved. Consequently, two displacement functions for the parts on the left ($v_1(x)$) and right ($v_2(x)$) side of the crack were obtained:

$$v_1(x) = 145.087 + 15.476 \cdot x + 0.05 \cdot x^2 - 8.333 \cdot 10^{-4} \cdot x^3 - 10 \cdot \mathrm{Ln}\left(2 \cdot 10^6 + 200000 \cdot x\right)$$
$$- x \cdot \mathrm{Ln}\left(2 \cdot 10^6 + 200000 \cdot x\right) \qquad 0\,\mathrm{m} \leq x \leq 5\,\mathrm{m}$$

$$v_2(x) = 145.060 + 15.481 \cdot x + 0.05 \cdot x^2 - 8.333 \cdot 10^{-4} \cdot x^3 - 10 \cdot \mathrm{Ln}\left(2 \cdot 10^6 + 200000 \cdot x\right)$$
$$- x \cdot \mathrm{Ln}\left(2 \cdot 10^6 + 200000 \cdot x\right) \qquad 5\,\mathrm{m} \leq x \leq 10\,\mathrm{m}$$

The quality of these solutions was verified by analysing the considered structure by implementing the COSMOS/M commercial finite element program. This model produced the midpoint's vertical displacement of -0.1009 m thus confirming excellent result from the simplified model which has produced the value of -0.1013 m (with 0.34 % discrepancy). Matching of the results between the two models was also very good for all other points along the beam as the discrepancy nowhere exceeded the value at the crack location.

The initial eigenfrequency estimation was generated by inserting bending solutions into Eqs.(1)-(3). This resulted in the value ω_1 = 18.53651 rad/s for the first eigenfrequency estimation. On the other hand, the 3D FE model produced the value of 18.53429 rad/s again confirming excellent quality of the result from the simplified model as the discrepancy between the two models' results was very low (0.01198 %).

Afterwards, also the buckling load P_{crit} was approximated from Eqs. (4)-(5) by implementing the same transverse displacements functions. The buckling load estimation was 258,229 N. Alternatively, the 3D FE model produced the value of 256,693 N thus showing that the simplified model produced the results with a moderately small discrepancy (0.5985 %).

Afterwards, general basic polynomial functions $w_1(x)$ and $w_2(x)$ were constructed by considering specific boundary conditions only (implementing zero boundary displacements as well as bending moments):

$$w_1(x) = 0.339 \cdot x - 8.328 \cdot 10^{-3} \cdot x^3 + 5.552 \cdot 10^{-4} \cdot x^4 \qquad 0\,\mathrm{m} \leq x \leq 5\,\mathrm{m}$$

$$w_2(x) = -8.194 \cdot 10^{-2} + 0.563 \cdot x - 8.328 \cdot 10^{-2} \cdot x^2 + 2.776 \cdot 10^{-3} \cdot x^3 \qquad 5\,\mathrm{m} \leq x \leq 10\,\mathrm{m}$$

The implementation of these functions into eigenfrequency computation resulted in the value of ω_1 = 18.6983 rad/s for the first eigenfrequency estimation which is clearly an inferior result to the value from the bending functions as discrepancy increased to a (still quite acceptable) value of 0.885 %.

The same basic polynomial displacement functions were further implemented in the buckling load analysis already producing an acceptable value for the buckling load: 259,238 N with a discrepancy of almost 1 % (which was again higher than at static bending functions' utilisation).

Since the displacement functions were simple polynomials, the upgrading of basic polynomial functions was afterwards separately performed for first eigenfrequency estimation as well as for buckling estimation without any mathematical issues, yielding the following functions:

$$w_{d,1}(x) = -0.204\cdot\omega^2 - 2.838\cdot10^{-2}\cdot\omega^2\cdot x - 4.436\cdot10^{-4}\cdot\omega^2\cdot x^2 - 3.777\cdot10^{-6}\cdot\omega^2\cdot x^3 + 1.889\cdot10^{-7}\cdot\omega^2\cdot x^4$$
$$+5.925\cdot10^{-8}\cdot\omega^2\cdot x^5 - 1.597\cdot10^{-9}\cdot\omega^2\cdot x^6 - 1.338\cdot10^{-10}\cdot\omega^2\cdot x^7 + 5.901\cdot10^{-12}\cdot\omega^2\cdot x^8$$
$$+8.873\cdot10^{-3}\cdot\omega^2\cdot(10+x)\cdot Ln(10+x) \qquad 0\,m \le x \le 5\,m$$

$$w_{d,2}(x) = 0.188\cdot\omega^2 + 2.7554\cdot10^{-2}\cdot\omega^2\cdot x + 4.192\cdot10^{-4}\cdot\omega^2\cdot x^2 - 3.419\cdot10^{-5}\cdot\omega^2\cdot x^3 + 1.624\cdot10^{-6}\cdot\omega^2\cdot x^4$$
$$+1.820\cdot10^{-8}\cdot\omega^2\cdot x^5 - 3.084\cdot10^{-9}\cdot\omega^2\cdot x^6 + 5.508\cdot10^{-11}\cdot\omega^2\cdot x^7$$
$$-8.122\cdot10^{-3}\cdot\omega^2\cdot(10+x)\cdot Ln(10+x) \qquad 5\,m \le x \le 10\,m$$

The newly derived at upgraded polynomial approximations $w_d(x)$ (obtained through four consecutive integrations of genuine basic polynomial functions) for the first eigenfrequency estimation produced the value ω_1 = 18.51594 rad/s, which has a rather low discrepancy (-0.099 %) against the 3D model value. However, it should be noted that the obtained value underestimated the value from the 3D model which is not consistent with the theory. Nevertheless, this divergence was a consequence of the computational model and not of the method, as the approximate method is being applied to a simplified model. It should be also noted that the upgrading process could have been further repeated. However, this was not executed due to the already low discrepancy achieved.

The separate upgrade of original polynomial function was executed also for the buckling problem. A new set of transverse displacements functions were derived at by realising that in buckling the transverse displacements are a sole function of axial compressive force P_{crit}. Therefore, the bending moments' functions were expressed as functions of applied axial force and transverse displacements. The considered problem's specific relation was $M_z(x) = -P_{crit}\cdot v(x)$. After two consecutive integrations the following functions were obtained:

$$w_{b,1}(x) = 1.208\cdot10^{-3}\cdot P_{crit} + 1.745\cdot10^{-4}\cdot P_{crit}\cdot x + 2.623\cdot10^{-6}\cdot P_{crit}\cdot x^2 - 1.157\cdot10^{-7}\cdot P_{crit}\cdot x^3 + 5.783\cdot10^{-9}\cdot P_{crit}\cdot x^4$$
$$-1.388\cdot10^{-10}\cdot P_{crit}\cdot x^5 - 5.246\cdot10^{-5}\cdot P_{crit}\cdot(10+x)\cdot Ln(10+x) \qquad 0\,m \le x \le 5\,m$$

$$w_{b,2}(x) = -1.934\cdot10^{-3}\cdot P_{crit} - 2.766\cdot10^{-4}\cdot P_{crit}\cdot x - 4.1845\cdot10^{-6}\cdot P_{crit}\cdot x^2 + 9.253\cdot10^{-8}\cdot P_{crit}\cdot x^3 - 1.157\cdot10^{-9}\cdot P_{crit}\cdot x^4$$
$$+8.410\cdot10^{-5}\cdot P_{crit}\cdot(10+x)\cdot Ln(10+x) \qquad 5\,m \le x \le 10\,m$$

With these two new functions the "strain" energy, Eq.(1), as well as the "applied" energy, Eq.(4), were re-evaluated. Finally, Eq.(5) yielded the improved value for the buckling load of P_{crit} = 257,606.5 N with a decreased discrepancy of 0.356 %.

In the last part, special alternative functions constructed from a dedicated polynomial solution were applied exclusively for the buckling analysis. These functions were constructed by additionally considering boundary information regarding shear forces which resulted in the unknown buckling load P_{crit} to be included in the displacement functions $w_s(x)$ (due to their complexity these functions are not presented here). The buckling load obtained from these functions was 258,346.2 N (with the discrepancy of 0.644 %). These functions produced the result which was better than the value obtained from the original basic general polynomial function, but worse from those from the improved general polynomial solution. Consequently, it was expected that the upgrading of these dedicated functions will result in the best approximation. However, the integration of these functions (that included the unknown buckling load P_{crit}) initially failed. Therefore, in the integrations within the upgrade process, the value of the unknown buckling load was taken as 258,346.2 N. Consequently, the bending functions become simple polynomials which allowed the integrations to be completed resulting in functions $w_{su}(x)$. The obtained buckling load was 257,557.4 N which became the simplified model's best results as the discrepancy was 0.3369 % (which is just slightly better than the value that resulted from the upgrading of basic general polynomials).

Afterwards, the above-described analyses were repeated for several locations of the crack along the beam, and the essential results are given in Tables 1 and 2.

It is obvious from Table 1 that almost all the simplified model's solution overestimate the corresponding "exact" values (i.e. values from the 3D FE model) as there are just two cases where the results just slightly underestimate the values from the 3D FE model. Initial simple general polynomial solutions $w(x)$ mostly provided the least accurate results. However, these functions allowed for upgrading ($w_d(x)$) that provided the situation's lowest values that, according to the theory, should also be the most accurate.

Similarly, Table 2 apparently shows that all the simplified model's solutions overestimate the corresponding values from the 3D FE model also in buckling analyses. Basic general polynomial solutions $w(x)$ initially provided results with less accuracy than the static bending displacement functions $v(x)$ for almost all locations. However, general polynomial solutions also allowed for the upgrade ($w_b(x)$) that in most case further produces slightly better results than the static bending displacement functions.

Furthermore, original special polynomial approximations $w_s(x)$ performed somehow better than general basic approximations. Ultimately, the best results for almost all locations were obtained from the upgrades ($w_{su}(x)$) of these special dedicated polynomials. The only exception is the case where the crack was 1 m from the weaker part of the structure where the static bending displacement functions produced just a slightly better result.

4.2 Cantilever, Clamped at the Right End

As the second structure, a cracked cantilever was examined. Again, derived at bending GDE's solutions $v(x)$ were compared against the 3D FE model solutions. The simplified model produced the free end's vertical displacement of -0.7272 m with a rather small discrepancy (0.032 %) against the 3D FE model result. However, it is interesting to note that the discrepancy at the crack location is slightly higher (0.142 %) as the discrepancy actually increased with the distance from the free end. Nevertheless, the general matching of the results between the two models was actually very good for all the points along the cantilever as the maximum discrepancy was everywhere below 1 %. After the verification of the simplified model's displacement functions, the initial eigenfrequency, as well as buckling load values were calculated and compared to the matching values from the 3D FE model. All these values are given in Tables 3 and 4.

After that, general basic polynomial functions $w(x)$ were constructed by considering example's specific boundary conditions only (considering zero boundary displacement and rotation as well as bending moment). These functions, as well as their upgrades ($w_d(x)$ and $w_b(x)$) , produced new values of the fundamental eigenfrequency and buckling load (see Tables 3 and 4).

The cantilever's study was completed by obtaining a buckling analysis dedicated polynomial solutions $w_s(x)$. These functions were constructed by considering additional boundary information regarding shear forces at the clamped end. In contrast to the simply supported beam structure, the inclusion of this additional information did not result in the unknown buckling load to be included in the newly derived at displacement functions. Consequently, the integration of these functions ($w_{su}(x)$) in the upgrading process did not cause any numerical problems. Both obtained values for the fundamental buckling load are given in Table 4.

It is evident from Tables 3 and 4 that static bending functions $v(x)$ produced a very decent result in the fundamental eigenfrequency estimation and, on the other hand, were quite unsuccessful in the buckling load analysis. Similarly, also general basic polynomials $w(x)$ performed well in dynamic analysis and were slightly less efficient in Euler load evaluation. Nevertheless, separate upgrades of basic polynomial approximations for both kinds of problems brought evident improvement of the results where the results for eigenfrequency once more exhibited slightly better agreement with the results from the 3D FE model. However, the special polynomial approximations $w_s(x)$ for buckling analysis already initially provided a decent result which was further efficiently improved with the upgrade process.

4.3 Propped Cantilever

All the above-described procedures were also repeated for the third structure, a propped cantilever with clamped-simply supported boundary conditions. The main results are summarised in Tables 5 and 6.

4.4 Clamped-clamped Beam

As the last a clamped-clamped beam was examined. The key results from the procedures already explained above are given in Tables 7 and 8.

5 Conclusions

The fundamental eigenfrequency, as well as Euler's critical load determination for transversely-cracked slender beams with a linear variation of width, was studied by implementing the simplified Okamura's computational model of cracked beams. The solutions for four beam structures were obtained through Rayleigh's energy method where kinetic and strain energy, as well as the work done by an external axial compressive force P, were evaluated by applying appropriate transverse displacement functions. In the paper, various displacements functions were applied. The results obtained with the implementation of the simplified model with the combination of various functions were afterwards compared to the results obtained from the pure numerical approach implementing 3D finite elements within the framework of the finite element method.

Initially, transverse displacements' functions $v(x)$ due to transverse load were implemented. Although they produced good values for the first eigenfrequency (with the discrepancy below 0.4 %) the quality of the results for the buckling load was not very consistent as for some cases they have produced very low discrepancies (0.6 %), but for some other examples, the discrepancy was evidently higher (up to 18 %). Furthermore, since these functions are not given as plain polynomials their upgrade through their integrations was not possible. Afterwards, alternative general polynomial functions $w(x)$ were constructed. Also these functions exhibited better results for eigenfrequency estimations. The maximum discrepancies were namely up to 3.8 % for eigenfrequency analysis and up to 10.8 % for buckling load analysis. However, their mathematical form allowed for integration and, therefore, the genuine polynomial functions were further upgraded, separately for eigenfrequency ($w_d(x)$) as well as for critical load ($w_b(x)$) estimation. These separated upgrades for eigenfrequency and buckling analyses have evidently improved the quality of the results. The discrepancies in eigenfrequency analysis almost vanished (below 0.1 %) while the discrepancies for the buckling load dropped below 1.5 %.

In the end, special polynomial functions $w_s(x)$ were constructed just for buckling analyses producing evidently better results than the general polynomial functions $w(x)$ with the maximum discrepancy around 2.1 %. These functions have been further upgraded. Although these improved functions ($w_{su}(x)$) generally produced the best results their improvement was not as apparent as in the previous cases as their discrepancies were already rather low prior to upgrading.

Despite the clear differences in the mathematical form and computational efforts between both computational models considered, the considered examples have thus shown that the application of the simplified model produces adequately matching of the results as no major differences are noticeable against 3D FE solutions. It can be thus concluded that the model is suitable for free vibration analyses with non-breathing crack as well as for buckling load evaluation. It is even reasonable to assume that by applying appropriate transverse displacement functions even higher eigenfrequencies could be evaluated.

The Okamura's computational model has thus proved itself to be usable for beams with linear variations of widths even by applying rather simple analysis methods. Nevertheless, it is rational to expect that by implementing more dedicated computational methods for eigenfrequency analysis as well as for buckling analysis this would also reflect in better results from the simplified model.

Acknowledgement

The author acknowledges the partial financial support from the Slovenian Research Agency (research core funding No. P2-0129 (A)).

References

[1] Bakhtiari-Nejad, F., Khorram, A., Rezaeian, M. (2014). Analytical estimation of natural frequencies and mode shapes of a beam having two cracks. *International Journal of Mechanical Sciences, 78*, 193–202.

[2] Biondi, B., Caddemi, S. (2007). Euler-Bernoulli beams with multiple singularities in the flexural stiffness". *European Journal of Mechanics - A/Solids, 26*(5), 789-809.

[3] Cao, M., Radzieński, M., Xu, W., Ostachowicz, W. (2014). Identification of multiple damage in beams based on robust curvature mode shapes. *Mechanical Systems and Signal Processing, 46*, 468–480.

[4] Greco, A, Pau, A. (2011). Detection of a concentrated damage in a parabolic arch by measured static displacements. *Structural Engineering & Mechanics, 39*(6), 751-765.

[5] Labib, A., Kennedy, D., Featherston, C. A. (2015). Crack localisation in frames using natural frequency degradations. *Computers and Structures, 157*, 51–59.

[6] Okamura, H., Liu, H.W., Chorng-Shin, C. (1969). A cracked column under compression. *Engineering Fracture Mechanics, 1*(3), 547-564.

[7] Palmeri, A., Cicirello, A. (2011). Physically-based Dirac's delta functions in the static analysis of multi-cracked Euler-Bernoulli and Timoshenko beams. *International Journal of Solids and Structures*, *48*(14-15), 2184-2195.

[8] Skrinar, M. (2009). Elastic beam finite element with an arbitrary number of transverse cracks. *Finite Elements in Analysis and Design, 45*(3), 181-189.

[9] Skrinar, M., Pliberšek, T. (2012). On the derivation of symbolic form of stiffness matrix and load vector of a beam with an arbitrary number of transverse cracks. *Computational material science, 52*(1), 253-260.

[10] Skrinar, M. (2013). Computational analysis of multi-stepped beams and beams with linearly-varying heights implementing closed-form finite element formulation for multi-cracked beam elements. *International Journal of Solids and Structures, 50*(14/15), 2527-2541.

[11] Skrinar, M., Imamovic, D. (2018). On the Bending Analysis of Multi-Cracked Slender Beams with Continuous Height Variations. *Periodica Polytechnica Civil Engineering*, *62*(4), 873-880. https://doi.org/10.3311/PPci.11897

Table 1: Results for fundamental eigenfrequency ω_1 [rad/s] for the simply supported beam

	Functions/model			
L_1	*v(x)*	*w(x)*	$w_d(x)$	3D FE
1 m	19.52986	22.49288	19.51193	19.48735
2 m	19.19155	20.77441	19.17580	19.14632
3 m	18.83349	19.65291	18.81885	18.78653
4 m	18.59319	18.99199	18.57628	18.58736
5 m	18.53651	18.69833	18.51594	18.53429
6 m	18.67056	18.70218	18.64743	18.62232
7 m	18.95363	18.93382	18.93022	18.91004
8 m	19.29570	19.29387	19.27301	19.25661
9 m	19.57118	19.62610	19.54853	19.53399

Table 2: Results for buckling load P_{crit} [N] for the simply supported beam

	Functions/model					
L_1	*v(x)*	*w(x)*	$w_b(x)$	$w_s(x)$	$w_{su}(x)$	3D FE
1 m	285104.3	339121.8	287056.7	308723.4	285468.7	283826.0
2 m	273286.9	294491.2	273529.8	282226.0	272810.7	271407.9
3 m	263088.2	271730.4	262402.4	266115.2	262117.1	260947.2
4 m	257902.7	261355.8	257111.3	258784.3	256987.0	255966.7
5 m	258229.1	259237.6	257606.5	258346.2	257557.4	256692.7

6 m	263352.1	263245.0	262925.0	263204.4	262921.3	262201.6
7 m	271791.8	271816.4	271538.2	271615.0	271524.6	270944.1
8 m	281167.4	282875.9	281011.4	281076.4	280880.7	280389.8
9 m	288346.3	292723.4	288132.0	288365.6	287851.1	287389.3

Table 3: Results for the fundamental eigenfrequency ω_1 [rad/s] of the cantilever

Method/model	ω_1	discrepancy
COSMOS 3D FE model	8.494559 rad/s	-
bending functions *v(x)*	8.519639 rad/s	0.295 %
general basic polynomial approximations *w(x)*	8.529249 rad/s	0.408 %
upgrade of general polynomial approximations $w_d(x)$	8.492690 rad/s	-0.022 %

Table 4: Results for buckling load P_{crit} [N] of the cantilever

Method/model	P_{crit}	discrepancy
COSMOS 3D FE model	76724.8 N	-
bending functions *v(x)*	90559.3 N	18.031 %
general basic polynomial approximations *w(x)*	81818.4 N	6.639 %
upgrade of basic polynomial approximations $w_b(x)$	76856.7 N	0.172 %
special polynomial approximations $w_s(x)$	78086.7 N	1.775 %
upgrade of special polynomial approximations $w_{su}(x)$	76809.6 N	0.110 %

Table 5: Results for the fundamental eigenfrequency ω_1 [rad/s] of the propped cantilever

Method/model	ω_1	discrepancy
COSMOS 3D FE model	27.95442 rad/s	-
bending functions *v(x)*	28.02343 rad/s	0.247 %
general basic polynomial approximations *w(x)*	29.02114 rad/s	3.816 %
upgrade of general polynomial approximations $w_d(x)$	27.94767 rad/s	-0.024 %

Table 6: Results for buckling load P_{crit} [N] of the propped cantilever

Method/model	P_{crit}	discrepancy
COSMOS 3D FE model	530869.9 N	-
bending functions *v(x)*	548504.2 N	3.322 %
general basic polynomial approximations *w(x)*	588154.4 N	10.791 %
upgrade of basic polynomial approximations $w_b(x)$	537350.8 N	1.221 %
special polynomial approximations $w_s(x)$	542066.3 N	2.109 %
upgrade of special polynomial approximations $w_{su}(x)$	533636.5 N	0.521 %

Table 7: Results for the fundamental eigenfrequency ω_1 [rad/s] of the clamped-clamped beam

method	ω_1	discrepancy
COSMOS 3D FE model	42.56036 rad/s	-
bending functions *v(x)*	42.72411 rad/s	0.385 %
general basic polynomial approximations *w(x)*	43.54528 rad/s	2.314 %
upgrade of general polynomial approximations $w_d(x)$	42.56100 rad/s	0.0015 %

Table 8: Results for buckling load P_{crit} [N] of the clamped – clamped beam

method	P_{crit}	discrepancy
COSMOS 3D FE model	1044774.0 N	-

bending functions *v(x)*	1101546.7 N	5.434 %
general basic polynomial approximations *w(x)*	1082479.2 N	3.609 %
upgrade of basic polynomial approximations $w_b(x)$	1030595.4 N	-1.357 %
special polynomial approximations $w_s(x)$	1052194.1 N	0.710 %
upgrade of special polynomial approximations $w_{su}(x)$	1028921.5 N	-1.517 %